answers for Dynamical Systems and Chaos, 11 April 2018.

I.

1. Fixed points: $x^* = 0, \sqrt{1-r}, -\sqrt{1-r}$. The last two exist when $r \leq 1$
2. Two new fixed point appear when $r_c = 1$.
3. $f'(x) = 1 - 2x - \frac{r}{(1+x)^2}$
4. $f'(x^* = 0) = 1 - r$, the fixed point becomes unstable for $r < 1$.

   $f'(x^* = \sqrt{1-r}) = 1 - 2\sqrt{1-r} - \frac{r}{(1+\sqrt{1-r})^2}$ is negative for < 1 and when $r \to 0$, $f' \to -1$

   $f'(x^* = -\sqrt{1-r}) = 1 + 2\sqrt{1-r} - \frac{r}{(1-\sqrt{1-r})^2}$ is negative for $0 < r < 1$ (and $r \to -\infty$ for $r \to 0$). So in this interval the fixed point is stable. $f'(x^* = -\sqrt{1-r})$ is positive and fixed point thus unstable for $r < 0$.

   It is A (super critical) pitchfork bifurcation around $r \approx 1$ and $r < 1$.
5. The fixed point $x^* = 0$ is stable for $r > 1$ (full line) and unstable for $r < 1$ (dotted line). The two other fixed points grow up as a square root for $0 < r < 1$ (full line). The 'negative' is unstable for $r < 0$ (dotted line).

II.

6. $$J = \left\{ \begin{array}{cc} -a * \cos^2(x) + a * \sin^2(x) - \cos(x)\cos(y) & \sin(x)\sin(y) \\ -\sin(x)\sin(y) & a * \sin^2(y) - a * \cos^2(y) + \cos(x)\cos(y) \end{array} \right\} \quad (1)$$
7. Nullclines: $\dot{x} = 0$ for $x = 0$, $x = \pi$ $y = \pi - x$. $\dot{y} = 0$ for $y = 0$, $y = \pi$ and $y = x$
8. Five fixed points in the interval: $(x^*, y^*) = (0,0), (0,\pi), (\pi,\pi), (\pi,0)$, $(x^*, y^*) = (\pi/2, \pi/2)$,
9. For $(x^*, y^*) = (0,0), (0,\pi), (\pi,\pi), (\pi,0)$, $\lambda = -2, 0$ (i.e. linearly marginal, but are really saddle-points)

   For $(x^*, y^*) = (\pi/2, \pi/2)$, $\lambda = 1 \pm i$ (i.e. complex unstable)
10. The flow circles in clock wise direction around the fixed point $(x^*, y^*) = (\pi/2, \pi/2)$. On x-axis flow goes in negative direction. On y-axis in the positive direction. As $t \to \infty$ the flow will approach a 'square' limit $x \in [0;\pi], y \in [0;\pi]$.
11. For $a = 0, \lambda = \pm i$, i.e. it is a (linear) center.

III.

12. Fixed points $x^* = 0, \pm\sqrt{\frac{-a}{1+a}}$. The first trivial fixed point exists always, the last two in the interval $-1 < a < 0$.
13. $f'(x_n) = \frac{1-x^2}{(1+x^2)^2} - a$
14. $f'(x^* = \pm\sqrt{\frac{-a}{1+a}}) = 2a + 2a^2 + 1$ and $0 < 2a + 2a^2 + 1 < 1$ if $-1 < a < 0$, therefore stable.
15. $f'(x^* = 0) = 1 - a = -1$ when $a_p = 2$
16. Assume $x_n = \epsilon$ where ϵ is small. Then $f(\epsilon) = \frac{\epsilon}{1+\epsilon^2} - 2\epsilon \approx -\epsilon$ and $f(-\epsilon) = -\frac{\epsilon}{1+\epsilon^2} + 2\epsilon \approx \epsilon$ thus a two-cycle.