Answers to Exam for Dynamical Systems and Chaos, 13 April 2016.

I.

1. Nullclines: $y = x, y = 2x - x^3$
2. Three fixed points: $(x^*, y^*) = (0,0)$, $(x^*, y^*) = (\sqrt{-1-a}, \sqrt{-1-a})$, $(x^*, y^*) = (-\sqrt{-1-a}, -\sqrt{-1-a})$.
3. The non-trivial fixed points appear at $a^* = -1$.
4. $$J = \left\{ \begin{array}{cc} a + 3x^2 & 1 \\ 1 & -1 \end{array} \right\} \quad (1)$$
5. For $(x^*, y^*) = (0,0)$, $\lambda = (a - 1 \pm \sqrt{a^2 + 2a + 5})/2$
   For $(x^*, y^*) = (\pm\sqrt{-1-a}, \pm\sqrt{-1-a})$, $\lambda = -a - 2 \pm \sqrt{a^2 + 2a + 2}$
6. In $a = -2$: For $(x^*, y^*) = (0,0), \lambda = (-3 \pm \sqrt{5})/2$, i.e. a stable node. For $(x^*, y^*) = (\pm\sqrt{-1-a}, \pm\sqrt{-1-a}), \lambda = \pm\sqrt{2}$, i.e. both are saddle points.
   In $a = 2$: For $(x^*, y^*) = (0,0), \lambda = \frac{1}{2} \pm \frac{\sqrt{13}}{2}$, i.e. a saddle point.
7. The trivial fixed point $(x^*, y^*) = (0,0)$ is unstable for $a > -1$, and becomes stable for $a < -1$. The nontrivial fixed points are unstable for $a < -1$. It is a sub-critical pitch-fork bifurcation.

II.

$$\ddot{x} + x + \epsilon h(x, \dot{x}) \;=\; 0 \quad , \quad a > 0 \quad h(x, \dot{x}) = (x^2 - a)\dot{x} + (\dot{x}^2 - b)x \quad , \quad a > 0 \quad , b > 0$$

8. $x_0 = r(T)cos(\theta), \dot{x}_0 = -r(T)sin(\theta)$
   $r'(T) = -r^3 < cos^2(\theta)sin^2(\theta) > +ar < sin^2(\theta) > +r^3 < cos(\theta)sin^3(\theta) > -br < cos(\theta)sin(\theta) >= -\frac{1}{8}r^3 + \frac{1}{2}ar$
9. $r'(T) = 0 \Rightarrow r^* = 0, r^* = 2\sqrt{a}$ $r^* = 0$ is unstable, $r^* = 2\sqrt{a}$ stable.
10. $a = 2 : r'(T) = dr/dT = \frac{1}{8}r(8 - r^2) \Rightarrow \frac{8dr}{r(8-r^2)} = dT \Rightarrow (\frac{1}{r} + \frac{r}{8-r^2})dr = dT \Rightarrow \ln r - \frac{1}{2}\ln(8 - r^2) = T + C$.
    Initial conditions $r(0) = 1$ results in: $C = -\frac{1}{2}\ln 7$. Inserting this gives after some algebra the solution $r(T) = \sqrt{\frac{8}{1+7e^{-2T}}}$. From this equation: $r(T) \to \sqrt{8}$.
11. $r\phi'(T) = -r^3 < cos^3(\theta)sin(\theta) > +ar < cos(\theta)sin(\theta) > +r^3 < cos^2(\theta)sin^2(\theta) > -br < cos^2(\theta) >= \frac{1}{8}r^3 - \frac{1}{2}br \Rightarrow \phi'(T) = \frac{1}{8}r^2 - \frac{1}{2}b$
12. $r^* = 2\sqrt{a} \Rightarrow \phi'(T) = \frac{1}{2}a - \frac{1}{2}b$
13. $\omega = 1 + \epsilon\phi' = 1 + \epsilon(\frac{1}{2}a - \frac{1}{2}b)$.

III.

14. $x_1^* = 0, x_2^* = \pi/2$
15. $f'(x_n) = 1 + a(\cos^2(x_n) - \sin^2(x_n))$.
16. $f'(\pi/2) = 1 - a \Rightarrow a_2 = 2$ $(f'(0) = 1 + a \Rightarrow a_1 = -2)$
17. $x_{n+1} \approx f(x_1) + (1 - 2(\cos^2(x_n) - \sin^2(x_n))_{x_1^*}(x_n - x_1) = 0 + (1-2)x_n = -x_n$ to linear order.
    Therefore putting in $x_0 = \epsilon \to x_1 = -\epsilon \to x_2 = \epsilon$. So to linear order it is a two-cycle.