Answers to Exam for Dynamical Systems and Chaos, 2 April 2014.

I.

1. $f(x) = 0 \quad \Rightarrow \quad x^* = 0, \pm\frac{1}{2}\sqrt{r(1-r)}$, the last two defined for $0 \le r \le 1$.
2. $f'(0) = r(r-1) \quad \Rightarrow \quad$ stable for $0 \le r \le 1$ otherwise unstable.
   $f'(\pm\frac{1}{2}\sqrt{r(1-1)}) = 2r(1-r)$, unstable for $0 \le r \le 1$.
3. Fixed point $x^* = 0$ bifurcates at $r_1 = 0$ and $r_2 = 1$ by Subcritical pitchfork bifurcation.
4. From below: Fixed point $x^* = 0$ (unstable) splits into to unstable point at $r_1 = 0$ and becomes itself stable. From above: Fixed point $x^* = 0$ (unstable) splits into to unstable point at $r_2 = 1$ and becomes itself stable. The two unstable branches connect in an ellipse.

II.

5. Fixed point: $(0,0)$. $\lambda = \pm i$, i.e. the fixed point is marginal.
6. The fixed point is marginal so we cannot conclude anything about stability from the linearized eigenvalues.
7. $\dot{V} = x\dot{x} + ay\dot{y} = xy + ay(-x - \epsilon y^3(1+x^2))$
8. $\dot{V} = -\epsilon y^4(1+x^2) < 0$ for $a = 1$. From Lyapunov theorem we can conclude $(0,0)$ is stable.
9. $\ddot{x} + x + \epsilon\dot{x}^3(1+x^2) = 0 \qquad x = r(T)cos\theta, \quad \dot{x} = -r(T)sin\theta$
   $h(x,\dot{x}) = -r^3 sin^3\theta(1 + r^2 cos^2\theta)$
   $r'(T) =< h(x,\dot{x})sin\theta) >= -r^3 < sin^4\theta > - \, r^5 < sin^4\theta cos^2\theta > \quad = -\frac{3}{8}r^3 - \frac{1}{16}r^5.$
10. As $r'(T) < 0 \quad \Rightarrow \quad r(T) \to 0$ as $T \to \infty$. The fixed point $(0,0)$ is stable as we found already.

III.

11. $x_n^* = f(x_n^*) \quad \Rightarrow \quad x_n^* = 0, \pm\sqrt{1 - \frac{1}{r}}$.
12. $f'(0) = r \quad \Rightarrow \quad$ stable for $r \in [-1, 1]$ otherwise unstable..
13. $f'(\pm\sqrt{1-\frac{1}{r}}) = -2r + 3 \quad \Rightarrow \quad$ unstable for $r < 0$ and for $r > 2$. Stable for $1 < r < 2$. The fixed point do not exist for $r \in]0, 1[$
14. The fixed points diverges to $\pm\infty$ as $r \to 0^-$.
15. $f'(0) = -1$ at $r_1 = -1$, $f'(0) = 1$ at $r_2 = 1$
    $f'(\pm\sqrt{1-\frac{1}{r}}) = -1$ at $r_3 = 2$
16. As the derivatives are -1, period doubling bifurcations take place at r_1 and r_3.
17. Fixed point $x_n^* = 0$ (stable) undergo a period-doubling at $r_1 = -1$, while it itself becomes unstable. Fixed point $x_n^* = 0$ goes from stable to unstable at $r_2 = 1$ where the two non-trivial fixed point are generated as stable fixed points. Both of these undergo a period doubling at $r_3 = 2$ while they both become unstable.