Answers: Exam for Dynamical Systems and Chaos, 10 April 2013.

I.

1. $x^* = 1 \pm \sqrt{1+2r}$
2. $r_c = -\frac{1}{2}$. Saddle-node bifurcation. Out of the blue.
3. $f'(x^*) = \pm\frac{1}{r}\sqrt{1+2r}$. For $r < 0$: '+' solution stable, '-' solution unstable.

   For $r > 0$: '+' solution unstable, '-' solution stable.
4. The stability of the fixed points changes as r passes through $r = 0$.

II.

5. x describes the prey and y describes the predator: x decreases when y is present, y increases when x is present.
6. Chaos is not possible in 2 dimensions (Poincare-Bendixon theorem, etc).
7. $(0,0), (\beta, \alpha)$
8. $\Delta = -\alpha * \beta < 0$, saddle point: $\lambda = \alpha, -\beta$, i.e. an unstable fixed point.
9. $\lambda_1 = i\sqrt{\alpha * \beta}, \lambda_2 = -i\sqrt{\alpha * \beta}$. The results yields a linear center. This is a borderline case and therefore not conclusive (p.137).
10.

$$\dot{E} = \frac{dE}{dx}\dot{x} + \frac{dE}{dy}\dot{y}$$

The two derivatives of E yields:

$$\frac{dE}{dx} = -y^{\alpha}e^{-y}x^{\beta}e^{-x} + y^{\alpha}e^{-y}\beta x^{\beta-1}e^{-x} = -E + \beta\frac{E}{x}$$

$$\frac{dE}{dy} = -y^{\alpha}e^{-y}x^{\beta}e^{-x} + \alpha y^{\alpha-1}e^{-y}x^{\beta}e^{-x} = -E + \alpha\frac{E}{y}$$

Inserting $\dot{x}$ and $\dot{y}$ yields:

$$\dot{E} = E((\frac{\beta}{x} - 1)(\alpha x - xy) + (\frac{\alpha}{y} - 1)(xy - \beta y)) = E * (0) = 0$$

Therefore E is a conserved quantity.

11. Nullclines: $x = \beta$ and $y = \alpha$. Rotation: Counter-Clockwise.

    As E is conserved, the second fixed point is therefore a non-linear center (Theorem 6.5.1 p.163). This means that both populations oscillate in numbers.

III.

12. The map has no intersection with the diagonal $x_{n+1} = x_n$ so there can be no fixed points.
13. If $x_0 = 1$ then $x_1 = 20, x_2 = 10, x_3 = 5, x_4 = 2.5, x_5 = 1.25$.

    If $x_0 = 2$ then $x_1 = 30, x_2 = 15, x_3 = 7.5, x_4 = 3.75, x_5 = 1.875$.

    Therefore $m = 5$.
14. $x_0 \to 10x_0 + 10 \to \frac{1}{2}(10x_0 + 10) \to \frac{1}{4}(10x_0 + 10) \to \frac{1}{8}(10x_0 + 10) \to \frac{1}{16}(10x_0 + 10) = x_0 \Rightarrow x_0 = \frac{5}{3}$. The cycles point are $x_0 = \frac{5}{3} \to x_1 = \frac{80}{3} \to x_2 = \frac{40}{3} \to x_3 = \frac{20}{3} \to x_4 = \frac{10}{3} \to x_5 = \frac{5}{3} = x_0$

    By the chain rule the stability is given by the product of the derivatives, i.e. $f^{5\prime} = 10 * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{10}{16} < 1$, i.e. the 5-cycle is stable.