Exam for Dynamical Systems and Chaos, 6 April 2011.
Duration: 3 hours. Books, notes and computers are allowed.
Exam consists of 3 exercises.
Answers can be written in Danish and English. Pencil is allowed
Results and solutions are posted on the home page.

I.

For the differential equation

$$\dot{x} = f(x) = r \ + x \ - \frac{1}{2}\ln(1+2x) \tag{1}$$

1. $f'(x) = 1 - \frac{1}{1+2x} = 0 \Rightarrow \tilde{x} = 0$ for all values of r
2. $f(\tilde{x}) = 0$ when $r_c = 0$
3. $ln(1+y) \simeq y - \frac{1}{2}y^2$ so $ln(1+2x) \simeq 2x - 2x^2$. Inserted gives $r + x - x + x^2 = 0 \Rightarrow x^* = \pm\sqrt{-r}$
4. Saddle node. Out of the blue.
5. $f'(+\sqrt{-r}) = 1 - \frac{1}{1+\sqrt{-r}} > 0$: unstable
   $f'(-\sqrt{-r}) = 1 - \frac{1}{1\sqrt{-r}} < 0$: stable
6. Bifurcation occurs at $r_c = 0$ and fixed point appear for $r < 0$. The upper branch ($x^* = +\sqrt{-r}$) dashed, linear in $-r$ as $r \to -\infty$; the lower branch ($x^* = -\sqrt{-r}$): full line, $x^* \to -\frac{1}{2}$ for $r \to -\infty$.

II.

$$\ddot{x} + x + \epsilon h(x, \dot{x}) \ = \ 0 \ \ , \ \ a > 0 \quad h(x, \dot{x}) = (x^2\dot{x}^2 - ax^2)\dot{x}$$

7. $x_0 = r(T)cos(\theta), \dot{x}_0 = -r(T)sin(\theta)$.
   Using averaging theory: $r'(T) = f(r) = ar^3 < cos^2(\theta)sin^2(\theta) > -r^5 < cos^2(\theta)sin^4(\theta) >= \frac{1}{8}ar^3 - \frac{1}{16}r^5$
8. $r'(T) = f(r) = \frac{1}{8}ar^3 - \frac{1}{16}r^5 = 0 \Rightarrow r^* = \sqrt{2a}$ or $r^* = 0$.
9. $f'(r^*) = \frac{3}{8}ar^{*2} - \frac{5}{16}r^{*4} = \frac{3}{4}a^2 - \frac{5}{4}a^2 = -\frac{1}{2}a^2 < 0$. Stable for all a.

III.

10. $x_n^* = 0, \frac{5}{7}$. All derivatives are $\pm\frac{5}{2}$ so they are all unstable.
11. All derivatives are $\pm\frac{5}{2}$ so Lyapunov exponent is $\ln\frac{5}{2}$.
12. All $x_0 \in]\frac{2}{5}; \frac{3}{5}[$ becomes > 1 after one iteration.
13. $x_0 = \frac{4}{25}$ hits $x_1 = \frac{2}{5}$ after one iteration and $x_0 = \frac{6}{25}$ hits $x_1 = \frac{3}{5}$ after one iteration so all $x_0 \in]\frac{4}{25}; \frac{6}{25}[$ becomes > 1 after two iterations.
    $x_0 = \frac{21}{25}$ hits $x_1 = \frac{2}{5}$ after one iteration and $x_0 = \frac{19}{25}$ hits $x_1 = \frac{3}{5}$ after one iteration so all $x_0 \in]\frac{19}{25}; \frac{21}{25}[$ becomes > 1 after two iterations.
14. $D = \ln(2)/\ln(5/2) \sim 0.76$