Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 11

March 9, 2015

10 One-Dimensional Maps

10.1 Fixed Points and Cobwebs

10.1.10

A fix point x^* should satisfy $1 + \frac{1}{2}\sin x^* = x^*$. Since $1 + \frac{1}{2}\sin x^* \in [1/2, 3/2]$, we then determine the intersections between $1 + \frac{1}{2}\sin x^*$ and x^* for $x^* \in [1/2, 3/2]$. It can be shown that $1 + \frac{1}{2}\sin x^*$ is strictly monotonous in this region. Since $1 + \frac{1}{2}\sin\frac{1}{2} \approx 1.240 > 1/2$ and $1 + \frac{1}{2}\sin\frac{3}{2} \approx 1.499 < 3/2$, there exists only one x^* such that $1 + \frac{1}{2}\sin x^* = x^*$. In other words, there is a unique fix point.

The derivative of f is $f'(x^*) = \frac{1}{2}\cos x^* < 1$. Therefore the fix point is stable.

10.1.12

(a) Note that $g(x) = x^2 - 4$ and $g'(x) = 2x$, we have

$$f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{x_n^2 - 4}{2x_n} = \frac{x_n^2 + 4}{2x_n}$$

(b) Solving $f(x^*) = x^*$ gives $x^* = \pm 2$.

(c) The derivative of f at the fix points are $f'(x^*) = 1/2 - 2/x^{*,2} = 0$. So it is superstable.

(d)

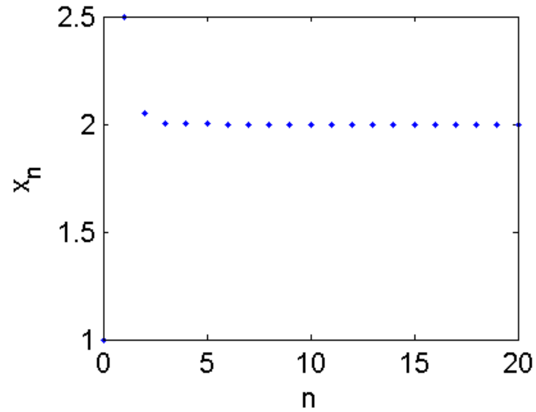

Figure 1: 10.1.12

10.3 Logistic Map: Analysis

10.3.2

(a) Since $f'(x) = r - 2rx$, solving $f'(x^*) = 0$ gives $x = 1/2$. Therefore, either p or q is 1/2. We don't need to check the other criterion since p and q guarantees a 2-cycle.

(b) First

$$f(f(x)) = f(rx(1-x)) = r^2x(1-x)\left[1 - rx(1-x)\right]$$

Since 1/2 is a solution for $f(f(x)) = x$, we have $r = 2, 1+\sqrt{5}$. Note that there exists a 2-cycle only when $r > 3$. Therefore, $r = 1+\sqrt{5}$.

10.3.10

(a) First notice that

$$x_{n+1} = \sin^2(\pi\theta_{n+1}) = \frac{1-\cos(2\pi\theta_{n+1})}{2} = \frac{1-\cos(2\pi(\theta_{n+1}+k))}{2}, \forall k \in \mathbb{Z}$$

so we have $x_{n+1} = \sin^2(\pi\theta_{n+1}) = \sin^2(2\pi\theta_n)$. Therefore,

$$4x_n(1-x_n) = 4\sin^2(\pi\theta_n)(1-\sin^2(\pi\theta_n)) = 4\sin^2(\pi\theta_n)\cos^2(\pi\theta_n) = \sin^2(2\pi\theta_n) = \sin^2(\pi\theta_{n+1}) = x_{n+1}$$

(b)

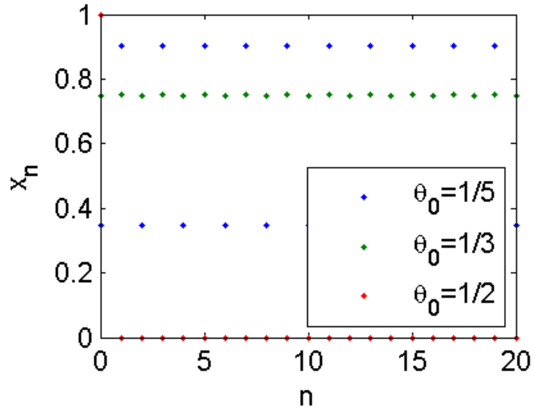

Figure 2: 10.3.10