Answers for Dynamical Systems and Chaos, 12 April 2023.

I.

1. $x^* = 0, \text{ or } x^* = 1 \pm \sqrt{1-r}$
2. A bifurcation occurs at $r_{c1} = 1$ in the point $x^* = 1$
3. $f'(x_n) = r - 4x_n + 3x_n^2$, $f'(0) = r$ so unstable for $r > 0$ and stable for $r < 0$.
   $f'(1+\sqrt{1-r}) = -2r+2+2\sqrt{1-r}$ so unstable for $r < 1$. $f'(1-\sqrt{1-r}) = -2r+2-2\sqrt{1-r}$ so stable for $0 < r < 1$ and unstable for $r < 0$.
4. Saddle-node bifurcation ("out-of-the blue").
5. It is a transcritical bifurcation.
6. $x^* = 0$ is unstable for $r > 0$ (dotted line), and stable for $r < 0$ (full line)
   $1 + \sqrt{1-r}$ is unstable for $r < 1$ (dotted line), $1 - \sqrt{1-r}$ stable for $0 < r < 1$ (full line) and unstable for $r < 0$ (dotted line).

II.

7. The fixed points are the origin ($r^* = 0$), which exists for any μ, γ. Then $(r^*, \theta^*) = (\sqrt{\mu}, \arcsin(-1/(\gamma\sqrt{\mu})))$ and $(r^*, \theta^*) = (\sqrt{\mu}, \pi - \arcsin(-1/(\gamma\sqrt{\mu})))$ which exist for $\gamma > 1/\sqrt{\mu}$ or $\gamma < -1/\sqrt{\mu}$
8. Considering that $(x, y) = (r\cos\theta, r\sin\theta)$, then $(\dot{x}, \dot{y}) = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta, \dot{r}\sin\theta + r\dot{\theta}\cos\theta)$, the system can be written in Cartesian coordinates as:

   $$\dot{x} = x(\mu - (x^2+y^2)) - y(1+\gamma y) \qquad \dot{y} = y(\mu - (x^2+y^2)) + x(1+\gamma y) \tag{1}$$

   which corresponds to:

   $$\dot{x} = \mu x - y + h.o. \qquad \dot{y} = x + \mu y + h.o. \tag{2}$$

   and therefore Jacobian:

   $$J = \left\{ \begin{array}{cc} \mu & -1 \\ 1 & \mu \end{array} \right\} \tag{3}$$

   with eigenvalues $\lambda = \mu \pm i$. Therefore, $r^* = 0$ turns from a stable spiral to an unstable spiral when $\mu = 0$.
9. When $\mu > 0$ and $\gamma = 1/2$ the only fixed point is $r^* = 0$ which is unstable. The trajectories around the origin are therefore repelled outwards. We can then look for which values of r, θ the trajectories point inward ($\dot{r} < 0$). This occurs when

   $$r(\mu - r^2) < 0 \qquad \Rightarrow \qquad r > \sqrt{\mu}.$$

   We can then for instance construct a trapping region around the origin (excluding the origin itself) with outer radius $r = 2\sqrt{\mu}$. All the trajectories point inside the trapping region, therefore there is a limit cycle.
10. Because the fixed point loses its stability (stable to unstable spiral) and a limit cycle appears around it, it's a supercritical Hopf bifurcation.

III.

11. $x_n^* = rx_n^* + {x_n^*}^3 \Rightarrow x_n^* = 0, x_n^* = \pm\sqrt{1-r}$. $x_n^* = 0$ exists always, $x_n^* = \pm\sqrt{1-r}$ exist for $r < 1$

12. $f'(x_n) = r + 3x_n^2$. $f'(0) = r$, stable for $r \in [r_-, r_+] = [-1, 1]$ and unstable for $r > 1$ and $r < -1$.

13. $f'(\pm\sqrt{1-r}) = 3 - 2r$ is unstable for $r < 1$

14. The fixed point $x_n^* = 0$ becomes stable for $r < 1$ while the two new fixed point are always unstable, a sub-critical pitch-fork bifurcation.

15. $f'(x^* = 0) = -1$ so a period doubling will take place.

16. $x_n^* = 0$ is stable in [-1,1] (full line) and unstable outside this interval (dotted line). $x_n^* = \pm\sqrt{1-r}$ are always unstable for $r < r_+ = 1$ (dotted lines).