Answers for Dynamical Systems and Chaos, 14 April 2021.

I.

1. Yes because $f(\theta + 2k\pi) = f(\theta)\ \forall k$
2. $(1 - \cos\theta^*) + (1 + \cos\theta^*)I = 0 \Rightarrow -I = tan^2(\theta^*/2) \Rightarrow \theta^* = 2tan^{-1}(\pm\sqrt{-I})$
3. The bifurcation occurs at I=0. For $I < 0$ there are two distinct fixed points, for $I > 0$ there are no fixed point. The two fixed points coalesce at $I = 0$. Therefore, it is a saddle-node bifurcation.
4. $f(\theta)' = \sin\theta - I\sin\theta = \sin\theta(1 - I)$ Since $I < 0$ when the two fixed points exist, the terms in parenthesis is always positive and f depends only on the sign of $\sin\theta$. So the angle in the upper quadrant is unstable and the one in the lower quadrant is stable.
5. $\frac{du}{dt} = \frac{d}{d\theta}\left[\tan\frac{\theta}{2}\right]\frac{d\theta}{dt} = \frac{1}{cos^2\frac{\theta}{2}}\frac{1}{2}\left[(1-\cos\theta) + (1+\cos\theta)I\right] = \tan^2\frac{\theta}{2} + I = u^2 + I$ which is the normal form of a saddle-node bifurcation on the line.
6. One has to solve the equation to find $u(t) = \sqrt{I}\tan(\sqrt{I}t)$ which exists if $I > 0$ and tends to infinite when t approaches $T_{blow} = \frac{\pi}{2\sqrt{I}}$
7. Rewrite the non-autonomous system into an autonomous one by setting $\dot{t} = 1$. Then of course for any value of a, $\dot{t} \neq 0$.

II.

8. For a gradient system: $(\dot{x}, \dot{y}) = -\nabla V \Rightarrow \int -\partial V/\partial x dx = x^2 + axy, \int -\partial V/\partial y dy = axy + by^2 \Rightarrow V(x,y) = x^2 + axy + by^2$
9. According to theorem for gradient systems it cannot have a closed orbit, that is only a fixed point which is $(x^*, y^*) = (0,0)$.
10.
$$J = \left\{ \begin{array}{cc} -2 & -a \\ -a & -2b \end{array} \right\} \quad (1)$$
11. $\lambda_\pm = -1 - b \pm \sqrt{1 - 2b + a^2 + b^2}, \qquad b = 1:\ \lambda_\pm = -2 \pm a$

    $-2 < a < 2:\ \ \tau < 0, \Delta > 0$: Stable node. $a > 2$ or $a < -2:\ \ \tau < 0, \Delta < 0$: Saddle point $a = \pm 2,\ \tau < 0, \Delta = 0$: One eigenvalue =0 (marginal), Non-isolated fixed points. $a = 0:\ \tau < 0, \Delta > 0$: Star (eigenvalues are equal = 2).
12. $a = \sqrt{7}$ and $b = 4$: $\lambda = -1, -9, \mathbf{v_1} = (\sqrt{7}, -1), \mathbf{v_2} = (1, \sqrt{7})$. Stable node, both eigendirections stable.
13. $a = 4$ and $b = 1$: $\lambda = 2, -6, \mathbf{v_1} = (1, -1), \mathbf{v_2} = (1, 1)$. Saddle point, first eigendirection unstable, second eigendirections stable.

III.

14. Fixed points: $x_n^* = \frac{\pi}{2}, \frac{3\pi}{2}$
15. $f'(x_n) = 1 - \frac{a}{2}\frac{\sin^3(x_n) + 2\cos^2(x_n)\sin(x_n)}{\sin^4(x_n)}$
16. $f'(\frac{\pi}{2}) = 1 - \frac{a}{2}, f'(\frac{3\pi}{2}) = 1 + \frac{a}{2}\ [a_{1,min}, a_{1,max}] = [0,4], [a_{2,min}, a_{2,max}] = [-4, 0]$
17. Superstable: $f'(\frac{\pi}{2}) = 1 - \frac{a}{2} = 0 \Rightarrow a_{ss} = 2$. Period doubling: $f'(\frac{\pi}{2}) = 1 - \frac{a}{2} = -1 \Rightarrow a_{PD} = 4$
18. $x_0 = x_1^* + \epsilon = \pi/2 + \epsilon \Rightarrow \sin(\pi/2 + \epsilon) \approx 1 - \frac{\epsilon^2}{2}, \cos(\pi/2+\epsilon) \approx -\epsilon$. Insert in $f: f(\pi/2 + \epsilon) = \pi/2 + \epsilon + 2\frac{-\epsilon}{1} = \pi/2 - \epsilon = x_1 \to \pi/2 - \epsilon + 2\frac{\epsilon}{1} = \pi/2 + \epsilon = x_2 = x_0$. That is a two-cycle to order ϵ.