Answers for Dynamical Systems and Chaos, 1 April 2020.

I.

1. $\dot{x} = \cot x \; - \; r \; \sin 2x = \cos x(\frac{1}{\sin x} - 2r \sin x) = 0 \Rightarrow \cos x = 0 \Rightarrow x^* = \frac{\pi}{2}$ or $x^* = \frac{3\pi}{2}$.

2. $f'(x,r) = -\frac{1}{\sin^2 x} - 2r \cos 2x = -\frac{1}{\sin^2 x} - 2r \cos^2 x + 2r \sin^2 x$. For both $x^* = \frac{\pi}{2}$ and $x^* = \frac{3\pi}{2}$ $f'(x^*, r) = 2r - 1$. Bifurcations takes place at $r_c = \frac{1}{2}$.

3. $f(x^*, r) = 0 \Rightarrow \sin^2 x^* = \frac{1}{2r}$, which has solutions when $r \geq r_c$

4. $f'(x^*, r) = -\frac{1}{\sin^2 x^*} + 4r \sin^2 x - 2r = 2 - 4r$. Below the bifurcation point $r < r_c$, $x^* = \frac{\pi}{2}$ $x^* = \frac{3\pi}{2}$ are both stable. Above the bifurcation point $r \geq r_c$ $x^* = \frac{\pi}{2}$ and $x^* = \frac{3\pi}{2}$ splits into two stable fixed points and become themselves unstable.

5. They are both pitchfork and super-critical.

6. $x^* = \frac{\pi}{2}$ is stable for $r < \frac{1}{2}$ and bifurcates into two stable fixed points for $r > \frac{1}{2}$ while it itself becomes unstable. One of the stable fixed points is just below $x = \frac{\pi}{2}$, the other is just above $x = \frac{\pi}{2}$

   $x^* = \frac{3\pi}{2}$ is stable for $r < \frac{1}{2}$ and bifurcates into two stable fixed points for $r > \frac{1}{2}$ while it itself becomes unstable.

   As $r \to \infty$ two fixed point both converges towards π the two others toward 0 (or 2π).

II.

7. $\nabla \cdot (g(x,y)(\dot{x}, \dot{y})) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \cdot (\frac{1}{y}(1 - (ax + y^2)), \frac{1}{x}(1 - (x + y)) = -\frac{a}{y} - \frac{1}{x} < 0$ always for $x > 0, y > 0$. So according to Dulac a limit cycle cannot exist.

8. Fixed points: $(x^*, y^*) = (0,0), (0,1), (\frac{1}{a}, 0), (2 - a, a - 1)$

9. $$J = \left\{ \begin{array}{cc} 1 - 2ax - y^2 & -2xy \\ -y & 1 - x - 2y \end{array} \right\} \qquad (1)$$

10. Eigenvalues: $(0,0) : \lambda = 1, 1$ unstable node; $(0,1) : \lambda = 0, -1$ stable node (but marginal in one direction) ; $(\frac{1}{a}, 0) : \lambda = -1, \frac{1}{3}$ (saddle point) ; $(2 - a, a - 1), a = \frac{3}{2}, \lambda = \frac{1}{8}(-5 \pm \sqrt{17}$ (stable node).

11. Nullclines: $y = 0, x = 0, y = 1 - x, y = \sqrt{1 - \frac{3}{2}x}$. $(0,0)$ has eigenvectors $(1,0), (0,1)$ which are unstable directions. $(0,1$ has eigenvectors $(1,-1)$(marginal),$(0,1)$ stable. $(\frac{1}{2}, \frac{1}{2})$ has eigenvectors $(\frac{1}{4}(1 + \sqrt{17}, 1), (\frac{1}{4}(1 - \sqrt{17}, 1)$. Basically all solutions will converge to this fixed point.

III.

12. $x_n^* = 2 - \frac{1}{r}$. $f'(x_n) = 1 - \frac{1}{2 - x_n}$ $f'(x_n^*) = 1 - r$

13. $f'((x_n^*) = 1$ for $r_{min} = 0, f'((x_n^*) = -1$ for $r_{max} = 2 \to [r_{min}, r_{max}] = [0, 2]$. Note that $x_n^* \to -\infty$ for $r \to 0^+$

14. Superstable: $f'(x_n) = 0 \to r_{ss} = 1$

15. Period doubling: $f'(x_n) = -1 \to r_{PD} = 2, x_n = x_{PD} = \frac{3}{2}$

16. $x_0 = \frac{3}{2} + \epsilon$. $\log(2(2 - \frac{3}{2} - \epsilon) = \log(1 - 2\epsilon) \approx -2\epsilon$. Therefore for the two-cycle: $x_0 = \frac{3}{2} + \epsilon \to x_1 = \frac{3}{2} + \epsilon - 2\epsilon = \frac{3}{2} - \epsilon \to x_2 = \frac{3}{2} - \epsilon + 2\epsilon = \frac{3}{2} + \epsilon = x_0$. Thus it is a two cycle.