Answers for Dynamical Systems and Chaos, 3 April 2019.

I.

1. $x^* = -1, \ln(-r)$

2. $f'(x) = r + 2\exp(x) + x\exp(x), f'(-1) = r + \exp(-1), f'(\ln(-r)) = -r - r\ln(-r)$
   $x^* = -1$ is stable for $r < -\exp(-1)$ and unstable for $r > -\exp(-1)$
   $x^* = \ln(-r)$ is unstable for $r < -\exp(-1)$ and stable for $r > -\exp(-1)$

3. $r_c = -\exp(-1)$. It is a transcritical bifurcation.

4. There is a change of stability at $r_c = -\exp(-1)$. The fixed point $x^* = \ln(-r)$ diverges as $r \to 0^-$.

II.

5. $(x_+^*, y_+^*) = (-\frac{1}{2} + \frac{1}{2}\sqrt{1+4a}, -\frac{1}{2} + \frac{1}{2}\sqrt{1+4a})$ $(x_-^*, y_-^*) = (-\frac{1}{2} - \frac{1}{2}\sqrt{1+4a}, -\frac{1}{2} - \frac{1}{2}\sqrt{1+4a})$

6.
$$J = \left\{ \begin{array}{cc} -1 & 1 \\ -\frac{a}{(1+x)^2} & -1) \end{array} \right\} \quad (1)$$

7. $\lambda_+ = -1 \pm 2\sqrt{\frac{-a}{(1+\sqrt{1+4a})^2}}$, $\lambda_- - = -1 \pm 2\sqrt{\frac{-a}{(1-\sqrt{1+4a})^2}}$. $b = -1$.

8. $\lambda_+ = -1 \pm 2i\sqrt{\frac{a}{(1+\sqrt{1+4a})^2}}$ In Strogatz terminology $\tau = -2 < 0$ and $\Delta > 0$ so it will always be stable. It is a stable spiral.

9. If you insert $a = -\frac{4}{25}$, then $\lambda_+ = -\frac{3}{2}, -\frac{1}{2}$ and $\tau = -2, \Delta = 3/4$ so it is a stable node. $\lambda_- = 1, -3$ and $\tau = -2, \Delta = -3$ so it is a saddle point.

10. For $a = -\frac{1}{4}$ the two fixed points collide in $(x^*, y^*) = (-\frac{1}{2}, -\frac{1}{2})$ and the eigenvalues are $\lambda = 0, -2$. It disappears for $a < -\frac{1}{4}$

III.

11. $\cos(x_n^*) = \frac{a}{b}$

12. $f'(x_n) = 1 + b * \sin(x_n) = 1 \pm b * \sqrt{1 - \cos^2(x_n)}$

13. $f'(x_n) = 1 \pm b * \sqrt{1 - \cos^2(x_n)} = 1 \pm \sqrt{b^2 - a^2} = 1 \Rightarrow a = \pm b$

14. $\cos(x_n^*) = \frac{a}{b} = 0 \Rightarrow x_n^* = \frac{\pi}{2}, \frac{3\pi}{2}$

15. Superstable: $f'(\frac{3\pi}{2}) = 1 - b = 0 \Rightarrow b_s = 1$

16. Period doubling: $f'(\frac{3\pi}{2}) = 1 - b = -1 \Rightarrow b_p = 2$