Answers for Dynamical Systems and Chaos, 6 April 2022.

I.

For the differential equation

$$\dot{x} = f(x) = \cos x \;-\; r\ \sin\ 2x = \cos x(1 - 2r \sin x) \tag{1}$$

1. $\dot{x} = 0 \Rightarrow x_1^* = \pi/2$ or $x_2^* = -\pi/2$ are always fixed points for any value of r.

2. $f'(x^*) = -\sin x^* + 2r\sin^2\ x^* - 2r\cos^2\ x^*$. $x_1^* = \pi/2$ stable for $r < 1/2$ and unstable for $r > 1/2$ $x_2^* = \pi/2$ stable for $r < -1/2$ and unstable for $r > -1/2$

3. $f'(\pi/2) \;=\; -1 \;+\; 2r \;=\; 0 \;\Rightarrow\; r_{c1} = 1/2$. $f'(-\pi/2) \;=\; 1 \;+\; 2r \;\Rightarrow\; r_{c2} = -1/2$.

4. $\dot{x} = 0 \;\Rightarrow\; \cos x^* - r\sin\ 2x^* \;=\; \;=\; \cos x^*(1 - 2r\sin\ x^*) \;=\; 0 \;\Rightarrow\; \sin\ x^* \;=\; \frac{1}{2r}$. $f'(x^*) = -\sin x^* \;+\; 2r sin^2 x^* \;-\; 2r\cos^2 x^* \;=\; -2r \;+\; \frac{1}{2r}$. The new fixed points exist only in the intervals $r \in [-\infty; -\frac{1}{2}]$ and $[\frac{1}{2}; \infty]$

5. $f'(x^*) = -\sin x^* + 2r\sin^2\ x^* - 2r\cos^2\ x^*$. Insert $\sin\ x^* \;=\; \frac{1}{2r}$ $f'(x^*) = -2r \;+\; \frac{1}{2r}$
   Fixed point $x_1^* = \pi/2$ stable for $r < \frac{1}{2}$ and for $r > \frac{1}{2}$ splits into two stable fixed points (full lines) and becomes unstable itself (dashed line). A supercritical pitchfork bifurcation.

   Fixed point $x_2^* = -\pi/2$ unstable for $r > -\frac{1}{2}$ and for $r < -\frac{1}{2}$ splits into two unstable fixed points (dashed lines) and becomes stable itself (full line). A subcritical pitchfork bifurcation.

6. Fixed point $x_1^* = \pi/2$ stable for $r < \frac{1}{2}$ and for $r > \frac{1}{2}$ splits into two stable fixed points (full lines) and becomes unstable itself (dashed line).

   Fixed point $x_2^* = -\pi/2$ unstable for $r > -\frac{1}{2}$ and for $r < -\frac{1}{2}$ splits into two unstable fixed points (dashed lines) and becomes stable itself (full line).

II.

7. Nullclines: $\dot{u} = 0 \rightarrow w = u - \frac{1}{3}u^3$ and $\dot{w} = 0 \rightarrow w = 1 + u$

8. Fixed point: $u - \frac{1}{3}u^3 + I = 1 + u \rightarrow u^3 = 3(I - 1)$

$$u^* = \sqrt[3]{3(I-1)}, w^* = 1 + \sqrt[3]{3(I-1)}$$

9. Stability of the fixed point. Jacobian:

$$J = \left\{ \begin{array}{cc} 1 - u^{*2} & -1 \\ 1/2 & -1/2 \end{array} \right\} \tag{2}$$

   $\tau = 1 - u^{*2} - \frac{1}{2} = \frac{1}{2} - u^{*2}, \Delta = \frac{1}{2}u^{*2}$. The fixed point is stable when $\tau < 0, \Delta > 0$. This happens when $u^* > \frac{1}{\sqrt{2}}$ or $u^* < -\frac{1}{\sqrt{2}}$, that is, when $I > \frac{1}{3}\frac{1}{2^{3/2}} + 1$ or $I < -\frac{1}{3}\frac{1}{2^{3/2}} + 1$. When I $= 4/3$, $u^* = 1, w^* = 2$ therefore $\tau = -1/2 < 0, \Delta = 1/2 > 0, \tau^2 - 4\Delta < 0$ then it is a stable spiral.

10. When I $= 1$ then the fixed point is $u^* = 0, w^* = 1$. Linearization predicts non isolated fixed points (as $\Delta = 0$) but we know from the previous analysis that there is only one.

11. We construct a box around the repeller as shown in figure. Given the information of the nullclines we know that on the upper edge of the rectangle the arrows point "south-west", on the right edge they point "north-west", on the lower edge they point "north-east" and on the left edge they point "south-east". Therefore in all cases they point toward the inside of the rectangle. Since the fixed point is a repeller, the trajectories will be repelled from there and we have constructed the surface that satisfies the Poincar-Bendixson theorem.

12. Given that a repeller turned into an attractor, in particular a stable spiral, surrounded by a limit cycle, it could be a supercritical Hopf Bifurcation.

III.

13. for $x \in [0, \frac{1}{3}] \Rightarrow x^* = 3x^* \Rightarrow x^* = 0$
for $x \in [\frac{1}{3}, 1] \Rightarrow x^* = \frac{3}{2}(1 - x^*) \Rightarrow x^* = \frac{3}{2} - \frac{3}{2}x^* \Rightarrow x^*(1 + \frac{3}{2}) = \frac{3}{2}x^* \Rightarrow \frac{5}{2}x^* = \frac{3}{2} \Rightarrow x^* = \frac{3}{5}$
Stability:
$f(x) = \frac{3}{2}(1 - x) \to f'(x) = \frac{3}{2}$, unstable since $|\lambda| = |f'(\frac{3}{5})| = \frac{3}{2} > 1$

14. Coweb: Draw from x-axes to graph to diagonal to graph ...etc

    Time series: On x-axes you have the discrete steps in n. On y-yxes the x_n values.

    (for diagrams, see below)

15. Location of fixed point info: Intersection of $x_{n+1} = f(x_n)$.
Stability info: The fixed point is unstable as absolute value of slope is largen than 1. Cobweb is "growing", i.e. the time-series grows away from the fixed point. We can also know the magnitude of instability (how unstable it is) from the slope of the tent in the first figure. -If the fixed point was stable, we would be able to extract information about its location from the time-series figure since the map would converge exponentially to this point. Therefore, it is true that the amount of features and information about the location depend on the stability of the fixed point.

16. • LL
$f^2(x) = x \Leftrightarrow f(f(x)) = x \Leftrightarrow f(3x) = x \Leftrightarrow 9x = x \Leftrightarrow x = 0$, fixed point. Therefore there is no limit cycle.
• RR
$f^2(x) = x \Leftrightarrow f(f(x)) = x \Leftrightarrow f(\frac{3}{2}(1-x)) = x \Leftrightarrow \frac{3}{2}(1 - \frac{3}{2}(1-x)) = x \Leftrightarrow \frac{3}{2}(1 - \frac{3}{2} + \frac{3}{2}x) = x \Leftrightarrow -\frac{3}{4} + \frac{9}{4}x = x \Leftrightarrow \frac{5}{4}x = \frac{3}{4} \Leftrightarrow x = \frac{3}{5}$, the other fixed point. Therefore there is no limit cycle.
• LR
$f^2(x) = x \Leftrightarrow f(f(x)) = x \Leftrightarrow f(3x) = x \Leftrightarrow \frac{3}{2}(1-3x) = x \Leftrightarrow \frac{3}{2} - \frac{9}{2}x = x \Leftrightarrow \frac{3}{2} = \frac{11}{2}x \Leftrightarrow x_1 = \frac{3}{11}$
$x_2 = f(\frac{3}{11}) = 3 \cdot \frac{3}{11} = \frac{9}{11}$
Therefore we have a 2-cycle.
• RL
$f^2(x) = x \Leftrightarrow f(f(x)) = x \Leftrightarrow f(\frac{3}{2}(1-x)) = x \Leftrightarrow 3(\frac{3}{2}(1-x)) = x \Leftrightarrow \frac{9}{2} - \frac{9}{2}x = x \Leftrightarrow \frac{9}{2} = \frac{11}{2}x \Leftrightarrow x_1 = \frac{9}{11}$
$x_2 = f(\frac{3}{11}) = \frac{3}{2}(1 - \frac{9}{11}) = \frac{3}{11}$
The same 2-cycle. Therefore we have only one 2-cycle.
The 2-cycle is unstable since:
$|f'(\frac{3}{11})||f'(\frac{9}{11})| = 3 \cdot \frac{3}{2} = \frac{9}{2} > 1$

17. Lyapunov exponent is given by:
$\lambda = \lim_{n\to\infty}(\frac{1}{n}\sum_{n=0}^{\infty} ln(|f'(x_i)|)) = \frac{1}{n}\frac{n}{3}ln|(3x)'| + \frac{1}{n}\frac{2n}{3}ln|(\frac{3}{2}(1-x))'|$
because they are uniformly distributed.
$\frac{1}{n}\frac{n}{3}ln|(3x)'| + \frac{1}{n}\frac{2n}{3}ln|(\frac{3}{2}(1-x))'| = \frac{1}{3}ln3 + \frac{2}{3}ln(\frac{3}{2}) = \frac{1}{3}ln3 + \frac{2}{3}ln3 - \frac{2}{3}ln2 = ln3 - \frac{2}{3}ln2,$