Exam for Dynamical Systems and Chaos, 3 April 2019.
Duration: 4 hours. Books, notes and computers are allowed.
All questions are equally weighted. Answers can be written in Danish and English.
Results and solutions are posted on the home page.

I.

Consider the differential equation where r is a real parameter

$$\dot{x} = f(x) = (x+1)(r+\exp(x)) \qquad (1)$$

1. Find the fixed points of (1).
2. Classify the fixed points according to their stability.
3. Determine the bifurcation point r_c. What type of bifurcation occurs at this point ?
4. Sketch the bifurcation diagram (fixed points versus r) and add the arrows of the flow (Full lines: stable fixed points, dashed lines: unstable fixed points).

II.

A protein x can repress itself by binding at its promotor at the DNA and produce the associated mRNA y leading to the coupled system of differential equations:

$$\dot{x} = y - x \quad , \quad \dot{y} = \frac{a}{1+x} - y \qquad (2)$$

where a is a real parameter.

5. Find the two fixed points (x_+^*, y_+^*) and (x_-^*, y_-^*) of (2) as a function of a (here + and - refer to the sign in front of the squareroot).
6. Derive the Jacobian matrix as a function of a.
7. Show that the eigenvalues of the fixed points can be written as $\lambda_+ = b \pm f_+(a)$ and $\lambda_- = b \pm f_-(a)$ and determine the constant b and the functions $f_+(a), f_-(a)$ (here subscripts + and - still refer to the two fixed points above).
8. Now consider the biological case $a > 0$ and the fixed point (x_+^*, y_+^*). Argue that the fixed point cannot become unstable. What type of fixed point is it?
9. We now allow $a \in]-\frac{1}{4}; 0]$. By either considering the eigenvalues or using Strogatz τ, Δ formalism, determine the type of the fixed points (x_+^*, y_+^*) and (x_-^*, y_-^*). (Hint: apply for instance the value $a = -\frac{4}{25}$).
10. What happens to the two fixed points at $a = -\frac{1}{4}$. Find the fixed points and the eigenvalues at this point.

III.

Consider a 1-d 'circle' map

$$x_{n+1} = f(x_n) = x_n + a - b * \cos(x_n) \qquad x_n \in [0; 2\pi] \qquad (3)$$

where a, b are real parameters and we consider $b > 0$.

11. Find a relation between the fixed points x^* and the parameters a, b.
12. Derive an expression for the derivative $f'(x_n)$ of the map (3).
13. Derive a relation between a and b where $f'(x^*) = 1$ (Hint: use a trigonometric relation between $\cos(x)$ and $\sin(x)$).
14. From now on set $a = 0$. Find the two fixed points x_1^*, x_2^* in this case (in the interval $[0; 2\pi]$).
15. Determine the value $b = b_s$ where one of the fixed points becomes superstable.
16. Determine the value $b = b_p$ where the same fixed point undergoes a period doubling bifurcation.