Exam for Dynamical Systems and Chaos, 11 April 2018.
Duration: 4 hours. Books, notes and computers are allowed.
All questions are equally weighted. Answers can be written in Danish and English.
Results and solutions are posted on the home page.

I.

Consider the one-dimensional differential equation where r is a real parameter:

$$\dot{x} \;=\; f(x) \;=\; x(1-x) - r\frac{x}{1+x} \tag{1}$$

1. Show that the system (1) can have one or three fixed points depending on the value of r. Determine these fixed points.
2. A bifurcation takes place at a value $r = r_c$. Determine r_c.
3. Derive the derivative $f'(x)$ of (1).
4. What type of bifurcation takes place at $r = r_c$? (Hint: Evaluate the stability of the fixed points for instance when $r \to 0$).
5. Draw the bifurcation diagram.

II.

A two-dimensional differential equation system, with a as a real parameter, is defined by:

$$\begin{aligned}\dot{x} &= -\sin(x)(a\cdot\cos(x) + \cos(y)) \\ \dot{y} &= \sin(y)(\cos(x) - a\cdot\cos(y)) \qquad x \in [0;\pi], y \in [0;\pi]\end{aligned} \tag{2}$$

6. Derive the Jacobian matrix as a function of a.
7. Now, unless otherwise stated, set $a = 1$. Find the nullclines for (2) and set arrows on them. Sketch the nullclines in the square $x \in [0;\pi], y \in [0;\pi]$. (Note: There are six nullclines and they are all straight lines).
8. Find the fixed points (x^*, y^*) for (2).
9. For the fixed points, find the eigenvalues and evaluate their stability. (Hint: Four of the fixed points have the same eigenvalues).
10. With an initial point away from the nullclines, sketch and describe the flow as $t \to \infty$ in the square $x \in [0;\pi], y \in [0;\pi]$.
11. Now set $a = 0$. What type of fixed point is now in the middle of the square.

III.

Consider a 1-d map

$$x_{n+1} = f(x_n) = \frac{x_n}{1+x_n^2} - a x_n \tag{3}$$

where a is a real parameter.

12. Find all fixed points x^* of (3) and determine in which intervals of a that they exist.
13. Derive an expression for the derivative $f'(x_n)$ of the map (3).
14. Determine the stability of the non-trivial fixed point in the parameter interval $a \in]-1;0[$.
15. Find the value $a = a_p$ where the trivial fixed point $x^* = 0$ undergoes period doubling.
16. Assume the value of x_n is small and positive $x_n \approx \epsilon$, then show to first order in ϵ that a two-cycle exist at $a = a_p$.